HOW TO DRAW CURVILINEAR PERSPECTIVE
4 - 5 - 6 VANISHING POINTS

PREVIEW

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Massimo Nicola Marrazzo

How to draw curvilinear perspective

4 - 5 - 6 vanishing points
How to draw curvilinear perspective
4 - 5 - 6 vanishing points

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Credits
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page 12, 20, 51, 59 furniture models http://www.sweethome3d.com
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ABBREVIATIONS AND SYMBOLS

CV = Center of Vision
D = Distance from Picture Plane
GL = Ground Line
GP = Ground Plane
H = Height point of view (eyes level)
HL = Horizon Line
▼ LMP = Left Measuring Point
⊕ LVP = Left Vanishing Point

MP = Measuring Point
Os = Observer
PP = Picture Plane
PS = Point of Station
PV = Point of View
▼ rMP = Right Measuring Point
⊕ rVP = Right Vanishing Point
VL = Vanishing Line
INTRODUCTION

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- Linear perspective (one, two, three-point perspective)
- Reflections
- Shadows

See the book (text in Italian):
"Prospettiva_ZeroSei", complete manual of perspective from 0 to 6 vanishing points"
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Chapter 1

4 - 5 - 6 vanishing points

Equirectangular Projection (360°)
The classical perspective, with straight lines. The maximum angle for the cone of vision is 60 degrees.

**Linear**

In this perspective, the horizontal lines are curved and the vertical are still straight lines. In the case of a complete cylindrical view, the horizontal width ranges from 180° to 360°.

**Cylindrical**

In the spherical perspective, horizontal and vertical lines are curves. The panorama is limited to vertical and horizontal angles of view of 180 degrees.

**Hemispherical**

With the projection onto the sphere, it is possible to draw a 360-degree panorama, also including the ceiling and the floor.

**Spherical**
**Linear perspective**

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**Curvilinear perspective**

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**Summary**

Based on Vanishing Points

- A 360-degree horizontal view and a 180-degree vertical view
- A 180-degree horizontal view and a 180-degree vertical view
- A 360-degree horizontal view and a 360-degree vertical view
4 vanishing points
Cylindrical perspective
Cylindrical perspective

In this perspective, the horizontal lines are curved.

The observer is inside a transparent cylinder. The surface of the cylinder intercepts the light rays that reach the Observer from the objects, determining a draw that, when unrolled and flattened, produces a draw with curved lines (curvilinear perspective).

Curvilinear perspective has a 360-degree field of view, with the disadvantage of curved lines, as is the case with very short focal length or wide-angle focal lenses (called fisheye).

Looking inside a cylinder, the curved lines of the curvilinear perspective would appear straight to us because turning the head is like being presented with infinite linear picture planes.

Objects in linear perspective, placed at the ends of the visual field, are seen stretched, an effect that does not occur in reality.

The human field of vision is considered curved; this is partly because the eye is spherical and partly because we rotate the head to look around.

Rotation of the head, with consequent vanishing points multiples.
Cylindrical perspective

The scene of vision is wider than 0°7: this enables the eye to draw the linear perspective without distortion.

Isodiametric within the boundaries of the visual field. The dimensions remain constant and increase the thinness of the objects seen foreshortened.

Frontal linear perspective

This is more 'natural human vision'. Going backwards, the size of objects is reduced.

Frontal curvilinear perspective
Cylindrical perspective

The sides are curved, due to the projection onto the Cylindrical Picture Plane.

Construction of an ellipse, in the closest cylindrical view.
Cylindrical perspective

The parallel lines projected on the cylindrical PL after the flat dimension (unrolling), draw a curve called sinuous.

Projection of parallel lines onto the cylindrical visual field.

Lines to be projected onto the cylinder.

Curved cylinder
- It represents the trend of height.
- Divided into n equal parts.

Cylinder unrolled onto a plane

The parallel lines projected on the cylindrical PL after the flat dimension (unrolling), draw a curve called sinuous.

Hemi-circle of the cylindrical PL.

Lenticular semi-circles.
To avoid excessive distortion, it is better to use only the perspective lines outside the cylindrical field of view, or use the spherical projection on the cylindrical picture plane, explained later in the chapter (as in rectangular projection).
5 vanishing points

Hemispherical perspective
Hemispherical field of vision

Projection of parallel lines onto the hemispherical surface.

Front view of the curves projected onto the hemisphere.

Projection of a cube onto the hemispherical field of vision.

Front view  Side view

Plan view
Hemispherical field of vision

5 vanishing points

The same cube is seen from two different perspectives.

Curvilinear perspective

Linear perspective
Comparison between the normal case of vision (90°) and the one hemispherical (180°).

This is what the observer sees inside the hemispheres of the field of view. The straight rays of the cube are now curved and depict a 180° view, much wider than linear perspective.
Hemispherical field of vision flattened

Vanishing Points

Curved Flatened

Unrolling of the hemisphere

Unrolling of the hemisphere onto the plane

To calculate the length of an arc:

\[ Y = r \cdot \frac{\theta}{180^\circ} \cdot \pi = \text{arc length} \]

\( r \) = radius of the circle
\( \theta \) = angle of the sector
\( \pi \approx 3.1415 \)
The divisions for horizontal angles correspond to the vertical ones.

Hemispherical field of vision

Hemisphere curved

Hemisphere flattened

Unrolling of the hemispheres onto the plane
Hemispherical field of vision CURVED
Grid 90°

Perspective diagram
Hemispherical field of vision CURVED
Grid 90°

Perspective construction

Plan view

Side view

Construction of ellipses

Curve is defined by 5 control points

Example
Hemispherical field of vision CURVED Grid 45°

Perspective diagram

Diagram of arrangement of this angle, in a curved hemispherical field of view.

Complete perspective diagram.
Hemispherical field of vision CURVED Grid 45°
Perspective construction

Construction of a rotated ellipse Curve is defined by 5 control points.

Construction ellipses by 3 given points through the 45° plane.

Complete perspective diagram.
Hemispherical field of vision CURVED Grid 30°

Perspective diagram

Diagram of arrangement of the angles, in a curved hemispherical field of vision.

Complete perspective diagram.
Hemispherical field of vision CURVED Grid 30°
Hemispherical field of vision FLATTENED
Grid 90°

Perspective diagram

Start by constructing a horizontal square grid with regular and constant divisions.

Take all the lines of arc, on a vertical plane, the points of the grid intersected by the semi-circumference. This mean forming the largest hemispherical field of vision.

With the new points, build an arc passing through 3 points, where the other 2 points are at the top.

It is possible find the basic line that converge towards the Ci, thanks by the interaction of the horizontal arc and the 90° arc.

In this way, the division of the basic line, after the flattening of the hemispherical field of view, are proportional and well-balanced.

So to have a regular translation for drawing, is necessary to make the division of the basic grid equal to each other.

Drawing an arc with 3 points

Have line with equal divisions

In this case, first drawing a base line with equal segments, and then draw the intersection with the 90° arc, obtain points to draw a circumference for 3 points, which represent the horizontal line of a grid projected onto the hemispherical field of view.
Hemispherical field of vision FLATTENED
Grid 45°
Perspective diagram

Diagram of arrangement of the angles, in a Hemispherical field of vision Flattened.

Complete perspective diagram.
Hemispherical field of vision FLATTENED Grid 45°

1. The 45° curve circles the vertical lines of the left wall, to find the points where the horizontal lines pass.

2. Horizontal lines of the left wall.

3. Vertical lines of the left wall, passing through the entrances of the fenestration grid.

a. Drawing the horizontal grid.
b. Join the ends at the horizontal grid (points 3),
   draw a series of curves by these points (points 1-8-3).
c. With the 45° line that cuts the vertical lines of the wall to the left, you can mark where the points pass through the horizontal lines of the left wall.

Plan view

Complete perspective diagram.
Hemispherical field of vision FLATTENED
Horizontal Grid 30° / 60°

The projected dimensions of the grid should be reported on the baseline.
Hemispherical field of vision FLATTENED
Horizontal Grid 30° / 60°

3

Vertical lines of the left wall, passing through the terminal points of the horizontal grid.

Mark where the 45° curve crosses these vertical lines of the left wall. The new horizontal curve passes through these points.

Horizontal lines of the left wall.

4

Complete perspective diagram.
Hemispherical field of vision FLATTENED
Horizontal Grid 30° / 60°
Tilted 20° downward

Finding a circle given 1 point

Axis on which all center of the circles, on the right side, are projected.

Note 1:
Convergence point of the horizontal lines to the left and tilted 20° downward.

The projected dimensions of the grid images should be matched on the baseline.
Hemispherical field of vision FLATTENED
Horizontal Grid 30° / 40°
Tilted 20° downward
Hemispherical field of vision CURVED

Ellipses

Frontal inclined planes
Hemispherical field of vision FLATTENED

Frontal inclined plane:

Circle through 3 plane points.

Frontal inclined solid:
Hemispherical field of vision PLATTENED

Turn the vanishing point pair to any angle, so you can draw related objects or inclined planes.
Construction, with the use of a compass, of a passing through a particular point.

1. Place point A.
2. Draw a line through point A.
3. Draw the perpendicular to the line.
4. Place the compass on point A and draw an arc passing through point A.

Horizontal curve

Vertical curve
Hemispherical field of vision FLATTENED
Gird 90°
Easy version

1. Projection towards CV

2. Construction of the circle, with the use of a compass

3. Compass center for 45° lines

4. The compass ends at the 45° points. The vertical divisions are different than those of the horizontal baselines

5. Compass centering for 45° lines
Hemispherical field of vision FLATTENED
Grid 30°/60°

Construction of circumferences with centers on the vertical measure line and a point passing through the 30° angle marked on the horizontal axis.

Construction of circumferences with centers on the vertical measure line and a point passing through the 120° angle marked on the horizontal axis.
Hemispherical field of vision FLATTENED

Grid 35°/60°

Circle by 3 points
1st and 3rd point are vertices of the vertical measure line 3 and the 3rd point is at the intersection of the basic grid square.

Dilate the vertical line with the same number of squares of the basic grid.

Dilate by 3 points
1st and 3rd point are vertices of the hemispheric angles and the 3rd point moves on the vertical line divided in plane smalls, corresponding to the number of the basic grid square.
6 vanishing points

Spherical perspective

6-point perspective
The difference between curved or flattened spherical perspective corresponds to the difference between world globe and planisphere:
in the first case, the drawing is represented on a three-dimensional sphere, in the second case, the drawing is projected onto a two-dimensional sheet.
This projection of the sphere on a two-dimensional plane involves the application of mapping techniques with different levels of distortion.
The unfolding and flattening of the sphere on the plane inevitably leads to distortions.
For this reason, it is preferred to project the spherical visual field on the cylinder, as this is easier to unroll on the sheet.
(See below Projection Equirectangular)
Projection of a cube on a spherical field of vision

Grid 90°

Cubic environment projected onto a spherical vision field and subsequent flattening on the sheet.
Projection of a cube on a spherical field of view

Cube visualization projected onto a spherical visual field and subsequent thinning on the sheet.
Spherical field of vision

Cubes projected on a spherical visual field

Meridians and parallels projected on spherical visual field

Egocentric pattern projected onto the spherical field of view and subsequent flattening on the sheet.

Pattern formed by meridians and parallels, projected onto the spherical field of view and subsequent flattening on the sheet.
Projection rules on a spherical field of vision split in half

Front hemispherical visual field

Rear hemispherical visual field

The 360° perspective with 6 operating points can be obtained from the union of two hemispherical views, with the benefit of a choice distinction for an easier reading.
6 vanishing points

Cylindrical perspective

6-point perspective - Equirectangular projections
The spherical field of vision is projected onto the cylindrical surface. The reason is that the cylinder is more easy to unfold on a sheet.

Equirectangular cylindrical projection

Unrolling a cylindrical surface onto a plane

Equirectangular Projection
Projection of the spherical field of vision onto the cylindrical surface:

1. Projections onto the spherical surface;
2. Projections from the spherical surface onto the cylindrical surface;
3. Unrolling of the cylindrical surface;
4. Unrolled surface.

The drawing on the sphere is consulted to flatten on a plane, but if you project it onto the cylinder, this can be opened and unrolled, generating a projection called Equiangular.

Rotation point to point of a spherical projection on the unrolled cylindrical surface.
Horizontal angle

Mark of contact with the visual field.

Diagonals with the feet inside the visual field.

The two parallel sides of the rectangles converge prospectively to the same Vanishing Point.
The grid inside the cylinder is represented in a distorted view; this is due to the process of representing also the space between the obstacle and the surface plane.
Construction of an intermediate wall

Distance of the intermediate wall from the central axis.
Construction of a Cube

The original shape of the cube is stretched.
Freehand construction of the equiangular projection

1. Projection from a spherical surface onto a cylindrical surface, tangent to the sphere.

2. Internal curves with 3 points.

3. Copy and move the same curve from A-B to B-C.

4. Drawing smooth curves through a set of points.

5. Continue drawing curves through points.

6. Complete perspective diagram.

Vertical lines drawn at the intersection with the points determined by the copied curve.

\[ r = \sqrt{180^2} \cdot a = \text{arc length} \]

- \( r \) = cylinder radius
- \( a \) = circumferential angle in radians on the plane. In this case, 295^\circ (semicircle)
- \( \pi = 3.1415 \)
A comparison between two types of cylindrical projection.
Chapter 2
Mathematical equations

The mathematical equations convert 3D coordinate \((x,y,z)\) to 2D \((x_1,z_1)\) coordinate.

Plotted with DecimalBASIC

Many thanks to Marco Masetti for the perspective formulas

PDF
https://marcomasettiiprospeettico.wordpress.com/2014/04/16/la-prospettiva-e-la-costruzione-dello-spazio-figurativo/

E-book
https://www.ibs.it/prospettiva-costruzione-dello-spazio-figurativo-ebook-marco-masetti/e/9788891154750

Paper book
https://www.amazon.it/prospettiva-costruzione-dello-spazio-figurativo/dp/8891130702
**Linear Perspective**  
(One-Two-Three-point perspective)

**Cylindrical Perspective**  
(Two-point perspective)  
(Four-point perspective)

**Cylindrical Perspective**  
(Six-point perspective)

**Hemispherical Perspective**  
(Five-point perspective)

**Spherical Perspective**  
(Six-point perspective)

Mathematical equations
Mathematical equations

Legend for the perspective diagrams

- \texttt{cos}: The cosine of \( x \)
- \texttt{atan}(y):\( y \)
- \( x = x \times \pi/180 \)
- \texttt{sign}(x): The sign of \( x \)
- \texttt{sqrt}: Square root of \( x \)

Note: In declarations, don't use \( \texttt{atan2}(y,x) \) or \( 90 \textdegree \texttt{sign}(x) \).

- Cartesian Coordinate System
- Spherical Coordinate System

Cubic grid projected on a spherical field of vision
Drawing perspective by mathematical equations

Math formulas

Linear perspective

$x, y, z$ = coordinates of point $P$

$x_1, y_1, z_1$ = transformed coordinates of point $P$

$x_1 = \frac{r}{f} \cdot x$

$y_1 = f$

$z_1 = \frac{r}{f} \cdot z$
Cylindrical perspective (curved)

\[ r_0 = \sqrt{x^2 + y^2} \]

- \( x, y, z \) = coordinates of point P
- \( x', y', z' \) = transformed coordinates of point P:
  - \( x' = r^2 \frac{x}{r_0} \)
  - \( y' = r^2 \frac{y}{r_0} \)
  - \( z' = z \)
Drawing perspective by mathematical equations

Unrolled Cylindrical View
2-vanishing point

Math formulas

Cylindrical perspective (unrolled)

\[ r = \text{radius of the cylinder} \]
\[ r_0 = \sqrt{x^2 + y^2} \]
\[ \alpha_l = \cos(y/r_0) \]
\[ x, y, z = \text{coordinates of point } P \]
\[ x_2, y_2, z_2 = \text{transformed coordinates of point } P \]

\[ x_2 = \text{sgn}(x) * r * \alpha_l * \pi / 180 \]
\[ y_2 = r \]
\[ z_2 = r * z / r_0 \]
Growing perspective by mathematical equations

Unrolled Cylindrical View
4-vanishing point

Math Formulas

Cylindrical perspective

$ r = $ radius of the cylinder
$ s = $ length of the cylinder
$ (x',y',z') = $ coordinates of point $ P$
$ (x,y,z) = $ transformed coordinates of point $ P$

$ x = r \cos(\theta) \cdot s$
$ y = \sqrt{r^2 - x^2}$
$ z = \frac{z'}{s}$
Spherical perspective (curved)

- $z_s = \frac{x^2+y^2+z^2}{2}$
- $x, y, z =$ coordinates of point $P$
- $x_s, y_s, z_s =$ transformed coordinates of point $P$

$$
\begin{align*}
z_s &= \pi^2 x/\pi z_s \\
y_s &= \pi^2 y/z_s \\
z_s &= \pi^2 z/z_s
\end{align*}
$$
Spherical perspective (flatted)

- radius of the sphere
- $r = \sqrt{x^2 + y^2 + z^2}$
- $x = \frac{r}{\sqrt{x^2 + y^2 + z^2}}$
- $y = \frac{y}{\sqrt{x^2 + y^2 + z^2}}$
- $z = \frac{z}{\sqrt{x^2 + y^2 + z^2}}$

$\delta = \text{sgn}(x) + \cos(\sqrt{x^2 + y^2 + z^2})$

$s, t, u, v$ = coordinates of point $P$

$s', t', u', v' = \text{transformed coordinates of point } P$

- $s' = \text{sgn}(x) + \text{sgn}(y)$
- $t' = t$
- $u' = \frac{r}{\sqrt{x^2 + y^2 + z^2}}$
- $v' = v$

Model projection:...
Spherical perspective
(flatness & vanishing point)

- \( r \) = radius of the sphere
- \( r' = \sqrt{r^2 - z'^2} \)
- \( a_1 = \text{cos}(z'/r) \)
- \( b_1 = \text{sin}(z'/r) \sqrt{1 - (z'/r)^2} \)
- \( x/y/z' = \text{coordinates of point } P \)
- \( x/y/z = \text{transformed coordinates of point } P \)

- \( x' = \text{cos}(\phi) \cdot r' \cdot \cos(\theta) \)
- \( y' = \text{sin}(\phi) \cdot r' \cdot \cos(\theta) \)
- \( z' = r' \cdot \sin(\theta) \cdot \text{cos}(\phi) \)
Drawing perspective by mathematical equations

Cylindrical perspective

- $r$: radius of the sphere
- $p = \sqrt{r^2 + y^2}$
- $p[i]$: polar coordinates of projected point $P$
- $p[j]$: two new coordinates, with $y$ for horizontal, and $x$ for vertical
- $x, y, z$: coordinates of point $P$
- $x, y, z$: transformed coordinates of point $P$
- $x3 = \arctan(y/x)$
- $y3 = \arccos(z/y)$

Front view

Top view

Flattened Spherical View

8-vanishing point

Math formulas
Drawing perspective by mathematical equations

Flattened Spherical View

6-vanishing point

Math formulas

Spherical perspective

$r =$ radius of the cylinder

$s$ = \sqrt{x^2+y^2+z^2}

$\theta = \arccos(y/r)$

$b_e = \text{sign}(x)*\arccos(\sqrt{(x^2+r^2)})$

$x, y, z$ = coordinates of point P

$x_1, y_1, z_1$ = transformed coordinates of point P

$x_1 = x^2*y^2/(y^2+r^2)$

$y_1 = r$

$z_1 = x^2*y^2/(y^2+r^2)$
Chapter 3

How to draw a rotated cube in 5-point perspective (angles 45°- 45°- 45°)
A 3-point perspective of a rotated cube: 45° - 45° - 45°

Rotation around the Z-axis by 45°   Rotation around the X-axis by 45°
A 3-point perspective of a rotated cube: 45°-45°-45°

Upper face

Top

Side

Front - Perspective

3D view
A 3-point perspective of a related cube: 45°-45°-45°

Upper face 1°
A 5-point perspective of a related cube: 45°-45°-45°

Upper face 2°
A 2-point perspective of a rotated cube: 45°-45°-45°
A 6-point perspective of a rotated cube: 45°-45°-45°

Front face
A 3-point perspective of a rotated cube: 45°-45°-45°

Drawing ellipses

1. Draw the Major and Minor axes
2. Diagonal line
3. Divide in 4 equal sections

4. Run diagonal lines across the sections
5. Draw a smooth curve through the 5 points

As an alternative to graphical construction, it is also possible to use plastic ellipse templates:

Major axis
Chapter 4

How to draw a rotated cube in 5-point perspective (angles 30°-60° & 15°-75°)
A 5-point perspective of a rotated cube: 30°-60°

Rotating a cube in three dimensions

Size of a rotated cube (30° - 60°)

Original size of a cube

Rotation around the Z-axis by 45°  Rotation around the X-axis by 45°

Curved Picture Plane

Front

Side

Front - Perspective

Top
A 6-point perspective of a rotated cube: 30°-60°

- Upper face
- Side
- Perspective
- Top
- 3D view

Extend edges till plane:

- Major axis

Curved picture plane:

- Major axis
A 5-point perspective of a rotated cube: 30°-60°

Front face

Plate

Extend edges till Plane

Front

Curved Picture Plane

Side

Perspective

Major axis

Top

3D view
A 3-point perspective of a rotated cube: 30°–90°

Right face
A 5-point perspective of a rotated cube: 15° - 75°

Rotating a cube in three dimensions

Size of a rotated cube (15° - 75°)

Original size of a cube

Curved Picture Plane

Front

Side

Front - Perspective

Top
Chapter 5

Curvilinear perspective:
“how to draw” and examples
horizontal grid

A horizontal line projected on the hemispherical field of view

As an alternative to graphical construction, it is also possible to use plastic ellipse templates.
horizontal grid

A grid of horizontal lines projected on the hemispherical field of view

vertical grid

A grid of vertical lines projected on the hemispherical field of view

Front

Top
How to find any point projected on the hemispherical field of view.
Heights and base points projected on the hemispherical field of view.

As an alternative to graphical construction, it is also possible to use plastic affine templates.
Lines inclined at 33° projected on a hemispherical field of view

1. Set and inclined plane
2. Draw perpendicular
3. Draw horizontal line 36
4. Intersect perpendicular from 4
5. Draw a line from 4 across plane of sphere

Graphical construction of an ellipse through points 1, 2, and 3.

Observing points of angle lines on the horizontal line of a hemispherical model field.

Top view
Lines inclined at 15° projected on a hemispherical field of view

The remaining axiom for a frontal object is on its circumference, not on the axis.
Lines inclined at 45° projected on a hemispherical field of view.
Objects in Two-point perspective projected on the hemispherical field of view.
Objects in Three-point perspective projected on the hemispherical field of view
A cube and a cylinder projected on a hemispherical field of view
Two geometric solids projected on a hemispherical field of view

The vanishing point for a frontal object is on the circumference, not on the axis.
Perspective grids
Chapter 6

Curvilinear perspective:
5 vanishing points with examples
Objects Lateral to the hemispherical field of vision

Front  Side

Objects Lateral to the hemispherical field of vision

Front  Side

Objects Lateral to the hemispherical field of vision

Front  Side

Objects Lateral to the hemispherical field of vision

Front  Side
Objects lateral to the hemispherical field of vision.
Objects Lateral to the hemispherical field of vision

Front

Side
Objects lateral to the hemispherical field of vision.
Objects Frontal to the hemispherical field of vision
Objects frontal to the hemispherical field of vision
Appendix
Perspective charts

Equirectangular Projection
Curvilinear perspective

Projection of a cube on the hemispherical field of vision

Grid 90°
CURVILINEAR PERSPECTIVE

Projection of a cube on the hemispherical field of vision

Grid 45°

Hemispherical field of vision flattened
(unfolding and flattening of this hemisphere on a sheet)

Hemispherical field of vision curved
(hemisphere curved 3D)

Hemispherical field of vision flattened
(unfolding and flattening of the hemisphere on a sheet)

Hemispherical field of vision curved
(hemisphere curved 3D)
CURVILINEAR PERSPECTIVE

Projection of a cube on the hemispherical field of vision

Grid 30°–60°

Hemispherical field of vision flattened
(unfolding and flattening of the hemispheres on a sheet)

Hemispherical field of vision curved
(hemispheres curved 3D)

Hemispherical field of vision flattened
(unfolding and flattening of the hemispheres on a sheet)

Hemispherical field of vision curved
(hemispheres curved 3D)
CURVILINEAR PERSPECTIVE
Perspective chart 30°/60°
CURVILINEAR PERSPECTIVE

Projection of a cube on the spherical field of vision

Grid 90°
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